Weighted Composition Semigroups on Hardy Spaces*

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ABSTRACT

The operator semigroups of the title are studied on Hardy spaces. Conditions are found for strong continuity, and the infinitesimal generator and its point spectrum are identified. A particular semigroup is used to study an averaging operator.

1. INTRODUCTION

The unit disk is denoted by $\mathbb{D}$, its boundary by $\partial\mathbb{D}$, and $H^p = H^p(\mathbb{D})$ denotes the usual Hardy space. A family $\{\varphi_t : t \geqslant 0\}$ is a (one-parameter) *semigroup* of analytic functions on $\mathbb{D}$ if it satisfies:

$$\text{for each } t \geqslant 0,\ \varphi_t : \mathbb{D} \to \mathbb{D} \text{ is analytic;} \tag{1.1i}$$

$$\varphi_0 \text{ is the identity map of } \mathbb{D}; \tag{1.1ii}$$

$$\varphi_t \circ \varphi_s = \varphi_{t+s} \text{ for } t, s \geqslant 0; \tag{1.1iii}$$

$$\varphi_t(z) \text{ is continuous in } (t, z) \text{ on } [0,\infty) \times \mathbb{D}. \tag{1.1iv}$$

Such a semigroup $\{\varphi_t\}$ induces a strongly continuous semigroup $\{T_t : t \geqslant 0\}$ of bounded linear operators on H^p, $1 \leqslant p < \infty$ [2], where for $t \geqslant 0$ the operator T_t is defined by

$$T_t(f) = f \circ \varphi_t, \qquad f \in H^p. \tag{1.2}$$

*A portion of this work was carried out while the author was a Ph.D. candidate, and is contained in the author's thesis submitted at the University of Illinois.

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If, further, $w: \mathbb{D} \to \mathbb{C}$ is analytic satisfying certain conditions (to be stated later), then the formula

$$S_t(f) = \frac{w \circ \varphi_t}{w} f \circ \varphi_t, \qquad t \geqslant 0, \quad f \in H^p, \tag{1.3}$$

also defines a semigroup $\{S_t\}$ of bounded operators on H^p.

We shall be concerned here with the weighted composition semigroups given by (1.3). The special case $w \equiv 1$ has been studied in [2]. We find conditions on the weight function w such that $\{S_t\}$ is strongly continuous. Then we identify the infinitesimal generator of $\{S_t\}$ and characterize its point spectrum. Finally we make use of one specific semigroup to study the operator J_p defined on H^p by

$$J_p(f)(z) = \frac{1}{1-z} \int_1^z f(\zeta) \frac{1}{1+\zeta} \, d\zeta. \tag{1.4}$$

The interest in J_p arises from the fact that it is an averaging operator which resembles, in some respects, the Cesàro operator studied in [8]. It also provides a nice illustration for the underlying theory of weighted composition semigroups.

It should be pointed out that systems of operators defined by (1.3) include the one-parameter groups of isometries of H^p studied in [1]. Such systems also are, in certain cases, the adjoints (on H^2) of the semigroups defined in (1.2).

We collect below, in the form of a theorem, some basic properties of the semigroups $\{\phi_t\}$ for future reference.

THEOREM 0. *Suppose $\{\varphi_t\}$ is a semigroup of analytic functions as in* (1.1). *Then*

(i) [2] *The limit $G(z) = \lim_{t \to 0} \partial \varphi_t(z)/\partial t$ exists uniformly on compact subsets of $\mathbb{D}$, and we have*

$$\frac{\partial \varphi_t(z)}{\partial t} = G(\varphi_t(z)), \qquad z \in \mathbb{D}, \quad t \geqslant 0.$$

The analytic function $G(z)$ is called the infinitesimal generator of $\{\varphi_t\}$.

(ii) [2] *The infinitesimal generator* $G(z)$ *has the unique representation*

$$G(z) = F(z)(\bar{b}z - 1)(z - b), \tag{1.5}$$

where $|b| \leqslant 1$ *and* F *is analytic with* $\operatorname{Re} F \geqslant 0$ *on* $\mathbb{D}$. *The distinguished point* b *in* (1.5) *is called the Denjoy-Wolff* (DW) *point of* $\{\varphi_t\}$.

(iii) [2] *Except for the case when* $\{\varphi_t\}$ *consists of elliptic Möbius transformations of* $\mathbb{D}$, *we have* $\lim_{t \to \infty} \varphi_t(z) = b$ *for each* $z \in \mathbb{D}$. *If* $|b| < 1$, *then* b *is a common fixed point of* φ_t, $t \geqslant 0$.

(iv) [3, 9] *If the DW point* b *is in* $\partial\mathbb{D}$, *then there is a unique univalent function* $h: \mathbb{D} \to \mathbb{C}$ *with* $h(0) = 0$, $h'(0) = 1$ *such that*

$$h(\varphi_t(z)) = h(z) + G(0)t, \qquad z \in \mathbb{D}, \quad t \geqslant 0. \tag{1.6}$$

(v) [3, 9] *If the DW point* b *is in* $\mathbb{D}$, *then there is a unique univalent function* $h: \mathbb{D} \to \mathbb{C}$ *with* $h(0) = 0$, $h'(0) = 1$ *such that*

$$(h \circ \gamma_b)(\varphi_t(z)) = e^{ct}(h \circ \gamma_b)(z), \qquad z \in \mathbb{D}, \quad t \geqslant 0, \tag{1.7}$$

where $c = G'(b)$ *and* $\gamma_b(z) = (z - b)/(1 - \bar{b}z)$.

The function h in either (1.6) or (1.7) will be called the *univalent function associated with* $\{\varphi_t\}$. The trivial semigroup, $\varphi_t(z) \equiv z$ for each $t \geqslant 0$, corresponds to the generator $G \equiv 0$.

We shall make use of the following lemma.

LEMMA 0 [5, p. 29]. *Suppose* $f \in H^p$ $(0 < p < \infty)$ *and* $\varphi: \mathbb{D} \to \mathbb{D}$ *is analytic. Then* $f \circ \varphi \in H^p$ *and*

$$\int_0^{2\pi} |f(\varphi(re^{i\theta}))|^p \, d\theta \leqslant \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta$$

for $0 < r < 1$.

2. STRONG CONTINUITY AND THE INFINITESIMAL GENERATOR

Suppose $\{\varphi_t\}$ is as in (1.1), with DW point b, and $w: \mathbb{D} \to \mathbb{C}$ is analytic with no zeros in $\mathbb{D} - \{b\}$. Then for f analytic on $\mathbb{D}$, the function $S_t(f)$

defined in (1.3) is analytic on $\mathbb{D}$ except possibly when $b \in \mathbb{D}$ and w has a zero at b. In this case $w(\varphi_t(z))/w(z)$ becomes analytic on $\mathbb{D}$ if we assign it the value $[\varphi_t'(b)]^n$ at b, where n is the order of the zero.

THEOREM 1. *Suppose* $0 < p < \infty$, $\{\varphi_t\}$ *is a semigroup of analytic functions with DW point* b, *and* $w: D \to \mathbb{C}$ *is analytic with no zeros in* $\mathbb{D} - \{b\}$. *Then either of the following two conditions implies that* $\lim_{t \to 0} \|S_t(f) - f\|_p = 0$ *for each* $f \in H^p$:

$$(\mathrm{C}_1) \qquad \limsup_{t \to 0} \left\| \frac{w \circ \varphi_t}{w} \right\|_\infty \leqslant 1;$$

$$(\mathrm{C}_2) \qquad w \in H^q \quad \textit{for some } q > 0 \quad \textit{and} \quad \limsup_{t \to 0} \left\| \frac{w \circ \varphi_t}{w} \right\|_\infty < \infty.$$

Proof. Under both conditions (C_1) and (C_2), $w(\varphi_t(z))/w(z) \in H^\infty$ for sufficiently small t. If $t > 0$ is arbitrary, there is a small δ and an integer n such that $t = n\delta$. The equation

$$\frac{w \circ \varphi_t}{w} = \prod_{k=1}^{n} \frac{w \circ \varphi_{k\delta}}{w \circ \varphi_{(k-1)\delta}}$$

then shows that $w(\varphi_t(z))/w(z) \in H^\infty$. Thus S_t is a bounded linear operator mapping H^p into H^p for each $t \geqslant 0$. If $f \in H^p$, from Lemma 0 we have

$$\|S_t(f)\|_p^p \leqslant \left\| \frac{w \circ \varphi_t}{w} \right\|_\infty^p \|f \circ \varphi_t\|_p^p \leqslant \left\| \frac{w \circ \varphi_t}{w} \right\|_\infty^p \left(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \right) \|f\|_p^p; \quad (2.1)$$

hence the set $\{\|S_t\| : t \in [0,1]\}$ is bounded.

Suppose condition (C_1) holds. We treat the case $1 < p < \infty$ first. Fix $p > 1$, $f \in H^p$, and a sequence $t_n \to 0$. Since H^p is a reflexive Banach space, the inequality (2.1) and the fact that $S_{t_n}(f)(z) \to f(z)$ for each $z \in D$ show that $S_{t_n}(f)$ converges weakly to f in H^p. From this and [4, Lemma II.3.27] we have

$$\|f\|_p \leqslant \liminf_{n \to \infty} \|S_{t_n}(f)\|_p.$$

Also

$$\begin{aligned} \limsup_{n \to \infty} \|S_{t_n}(f)\|_p &\leqslant \limsup_{n \to \infty} \left\| \frac{w \circ \varphi_{t_n}}{w} \right\|_\infty \lim_{n \to \infty} \left(\frac{1 + |\varphi_{t_n}(0)|}{1 - |\varphi_{t_n}(0)|} \right)^{1/p} \|f\|_p \\ &= \|f\|_p. \end{aligned}$$

It follows that $\|S_{t_n}(f)\|_p \to \|f\|_p$ as $n \to \infty$. This together with the weak convergence of $S_{t_n}(f)$ to f and the uniform convexity of H^p for $1 < p < \infty$ give the desired conclusion: $\lim_{n \to \infty}\|S_{t_n}(f) - f\|_p = 0$. Next let $0 < p \leqslant 1$. In this case the metric on H^p is $\|f - g\|_p^p$. Fix q, $1 < q < \infty$, and $g \in H^q$, and consider the triangle inequality

$$\|S_t(f) - f\|_p^p \leqslant \|S_t(f) - S_t(g)\|_p^p + \|S_t(g) - g\|_p^p + \|g - f\|_p^p.$$

We have $\|S_t(f) - S_t(g)\|_p^p \leqslant \|S_t\|^p\|f - g\|_p^p$ and $\|S_t(g) - g\|_p^p \leqslant \|S_t(g) - g\|_q^p$. Thus for $t \in [0,1]$ we have

$$\|S_t(f) - f\|_p^p \leqslant (M+1)\|f - g\|_p^p + \|S_t(g) - g)\|_q^p$$

where M is a common bound of $\|S_t\|^p$ for $t \in [0,1]$. From this inequality and the first part of the proof, together with the fact that H^q is dense in H^p, we get the desired conclusion.

Assume now that condition (C_2) holds. Since $\{\|S_t\| : 0 \leqslant t \leqslant 1\}$ is bounded, and the polynomials are dense in H^p, it suffices to show that for each polynomial P, $\lim_{t \to 0}\|S_t(P) - P\|_p = 0$. This is equivalent to showing that $\lim_{t \to 0}\|S_t(\chi_n) - \chi_n\|_p = 0$ for each $n \geqslant 0$, where $\chi_n(z) = z^n$. We shall take $n = 1$ (the proof for other values of n is similar). Suppose, by way of contradiction, that there is a sequence $\{t_k\}$ tending to zero such that

$$\left\|\frac{w \circ \varphi_{t_k}}{w}\varphi_{t_k} - \chi_1\right\|_p \geqslant \alpha > 0 \qquad \text{for} \quad k = 1,2,3,\ldots.$$

Choosing $w \equiv 1$ in the proof under condition (C_1), we have $\lim_{t \to 0}\|f \circ \varphi_t - f\|_p = 0$ for each $f \in H^p$. In particular

$$\|\varphi_{t_k} - \chi_1\|_p \to 0 \qquad \text{as} \quad k \to \infty,$$

so there is a subsequence $\{t_{k_n}\}$ such that

$$\varphi_{t_{k_n}}(e^{i\theta}) \to e^{i\theta} \qquad \text{a.e. on } \partial\mathbb{D},$$

where $\varphi_t(e^{i\theta})$ denotes the boundary function of φ_t. Now $w \in H^q$ with $q > 0$; therefore

$$\lim_{n \to \infty}\|w \circ \varphi_{t_{k_n}} - w\|_q = 0;$$

hence there is a further subsequence, which by a change of notation we may assume is $\{t_m\}$, such that

$$(w \circ \varphi_{t_m})(e^{i\theta}) \to w(e^{i\theta}) \qquad \text{a.e. on } \partial\mathbb{D}.$$

It follows that the boundary function $b_{t_m}(e^{i\theta})$ of $w(\varphi_t(z))/w(z)$ satisfies $b_{t_m}(e^{i\theta}) \to 1$ a.e. on $\partial\mathbb{D}$. From this,

$$b_{t_m}(e^{i\theta})\varphi_{t_m}(e^{i\theta}) - e^{i\theta} \to 0 \qquad \text{a.e. on } \partial\mathbb{D}.$$

Also $|b_{t_m}(e^{i\theta})\varphi_{t_m}(e^{i\theta}) - e^{i\theta}| \leqslant \|w \circ \varphi_t / w\|_\infty + 1$, so we can apply the bounded-convergence theorem to find

$$\int_0^{2\pi} \left| b_{t_m}(e^{i\theta})\varphi_{t_m}(e^{i\theta}) - e^{i\theta}\right|^p d\theta \to 0 \qquad \text{as} \quad m \to \infty.$$

It follows that

$$\left\| \frac{w \circ \varphi_{t_m}}{w}\varphi_{t_m} - \chi_1 \right\|_p \to 0 \qquad \text{as} \quad m \to \infty,$$

which is a contradiction to the original choice of $\{t_k\}$. ■

Remark. We have not been able to decide whether conditions (C_1) and (C_2) are independent of each other.

Theorem 1 establishes the strong continuity of $\{S_t\}$. The next theorem gives the infinitesimal generator.

Theorem 2. *Suppose $1 \leqslant p < \infty$ and $\{\varphi_t\}$ is a semigroup of analytic functions with infinitesimal generator G and DW point b. Also let $w : \mathbb{D} \to \mathbb{C}$ be analytic with no zeros in $\mathbb{D} - \{b\}$, and suppose that condition (C_1) or (C_2) of Theorem 1 is satisfied. Then the infinitesimal generator Δ_p of $\{S_t\}$ on H^p has domain $\mathscr{D}(\Delta_p) = \{f \in H^p : Gf' + w'\frac{G}{w}f \in H^p\}$, and*

$$\Delta_p(f) = Gf' + w'\frac{G}{w}f \tag{2.2}$$

for each $f \in \mathscr{D}(\Delta_p)$.

Proof. By definition, the domain of Δ_p is

$$\mathscr{D}(\Delta_p) = \left\{ f \in H^p \colon \lim_{t \to 0} \frac{S_t(f) - f}{t} \text{ exists in } H^p \right\}.$$

Let also

$$\mathscr{D} = \left\{ f \in H^p \colon Gf' + w' \frac{G}{w} f \in H^p \right\}.$$

It is clear that $\mathscr{D}$ is a linear manifold in H^p. For $z \in \mathbb{D}$, the pointwise limit of $[S_t(f)(z) - f(z)]/t$ as $t \to 0$ is

$$\frac{\partial}{\partial t}\left(\frac{w(\varphi_t(z))}{w(z)} f(\varphi_t(z)) \right) = G(z) f'(z) + w'(z) \frac{G(z)}{w(z)} f(z).$$

Since convergence in H^p implies in particular pointwise convergence, it follows that $\mathscr{D}(\Delta_p) \subseteq \mathscr{D}$, so the operator Δ defined on $\mathscr{D}$ by $\Delta(f) = Gf' + w'(G/w)f$ extends Δ_p. The assertion of the theorem is trivial if $\{\varphi_t\}$ is the trivial semigroup; thus assume $G \not\equiv 0$. Applying [4, Lemma VIII.1.4 and Theorem VIII.1.11], we find that the limit

$$\omega_0 = \lim_{t \to \infty} \frac{\log \|S_t\|}{t}$$

exists, $-\infty \leqslant \omega_0 < \infty$, and if $\operatorname{Re} \lambda > \omega_0$ then $\Delta_p - \lambda$ has a bounded inverse on H^p. In particular if $\operatorname{Re} \lambda > \omega_0$ then $\Delta_p - \lambda$ is onto H^p.

Suppose first that $\{\varphi_t\}$ has DW point on $\partial \mathbb{D}$. By Theorem 0(iv), $\varphi_t(z) = h^{-1}(h(z) + G(0)t)$, where $h \colon \mathbb{D} \to \mathbb{C}$ is analytic and univalent, and an easy computation shows that $G(z) = G(0)/h'(z)$. Pick $\lambda > \max(\omega_0, \log\|S_1\|)$. We show that $\Delta - \lambda$ is one-to-one. If not, then there is $f \in H^p$ such that

$$\frac{G(0)}{h'} f' + w' \frac{G(0)}{h'w} f = \lambda f.$$

The only possible analytic solutions of this are

$$f(z) = c \frac{1}{w(z)} \exp\left(\frac{\lambda}{G(0)} h(z) \right) \tag{2.3}$$

where c is a constant. From this and (1.6) we find $S_t(f)(z) = e^{\lambda t}f(z)$, so $e^{\lambda t}\|f\|_p \leqslant \|S_t\|\,\|f\|_p$. If f is not identically zero, then by taking $t = 1$ in the above inequality we arrive at a contradiction to the choice of λ. Thus $\Delta - \lambda$ is one-to-one. Since it also extends $\Delta_p - \lambda$, which is onto H^p, it follows that $\mathscr{D}(\Delta_p) = \mathscr{D}$ and $\Delta_p = \Delta$.

Next assume that $\{\varphi_t\}$ has DW point $b \in \mathbb{D}$. In this case we find $G(z) = G'(b)h_0(z)/h_0'(z)$ where $h_0 = h \circ \gamma_b$ [the notation of Theorem 0(v)]. If λ is a complex number and f is analytic on $\mathbb{D}$ with $f \not\equiv 0$ and $\Delta(f) = \lambda f$, then pick r such that $|b| < r < 1$ and f has no zeros on $|z| = r$. We have

$$G'(b)\frac{h_0}{h_0'}f' + w'\frac{G'(b)}{w}\frac{h_0}{h_0'}f = \lambda f,$$

so

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{f'(z)}{f(z)}\,dz = \frac{\lambda}{G'(b)}\frac{1}{2\pi i}\int_{|z|=r}\frac{h_0'(z)}{h_0(z)}\,dz - \frac{1}{2\pi i}\int_{|z|=r}\frac{w'(z)}{w(z)}\,dz. \tag{2.4}$$

From this and the argument principle it follows that the set of eigenvalues of Δ is countable, so we can find $\lambda > \omega_0$ such that $\Delta - \lambda$ is one-to-one. As in the previous case, it follows that $\mathscr{D}(\Delta_p) = \mathscr{D}$ and $\Delta_p = \Delta$. ∎

COROLLARY 1. *Suppose* $1 \leqslant p < \infty$, *and* $\{\varphi_t\}$ *and* w *satisfy the hypotheses of Theorem* 1. *Then* $\{S_t\}$ *is continuous in the uniform operator topology of* H^p *if and only if* $\{\varphi_t\}$ *is the trivial semigroup.*

Proof. Assuming that the infinitesimal generator Δ_p of $\{S_t\}$ is bounded on H^p, we conclude that $\Delta_p(f) = Gf' + w'(G/w)f$ for each $f \in H^p$. Taking f to be a constant, we find that $w'G/w \in H^p$. Next take f to be the function $\chi_1(z) = z$ to find $G + w'(G/w)\chi_1 \in H^p$, so $G \in H^p$. By taking f to be χ_n, $\chi_n(z) = \chi^n$, $n = 1,2,\ldots$, we find $\|nG + w'(G/w)z\|_p \leqslant \|\Delta_p\|$. Then

$$\left\| n\|G\|_p - \left\| w'\frac{G}{w}z\right\|_p \right\| \leqslant \|\Delta_p\| \qquad \text{for} \quad n \geqslant 1,$$

so $G \equiv 0$. ∎

The following theorem characterizes the point spectrum $\pi(\Delta_p)$ of Δ_p, in terms of the univalent function h associated with $\{\varphi_t\}$.

THEOREM 3. *Let $1 \leqslant p < \infty$, and let $\{\varphi_t\}$ be a semigroup of analytic functions mapping $\mathbb{D}$ into $\mathbb{D}$ with infinitesimal generator $G(z)$, DW point b, and associated univalent function h. Also let $w: \mathbb{D} \to \mathbb{C}$ be analytic with no zeros in $\mathbb{D} - \{b\}$ and satisfy either of the conditions of Theorem 1.*

(i) *Suppose $b \in \mathbb{D}$. Then $\pi(\Delta_p) \subseteq \{kG'(b): k = 0, 1, 2, \ldots\}$. The point $kG'(b)$ is in $\pi(\Delta_p)$ if and only if $(1/w)(h \circ \gamma_b)^k \in H^p$. Moreover each eigenspace is one-dimensional.*

(ii) *Suppose $b \in \partial\mathbb{D}$. Then for $\lambda \in \mathbb{C}$, $\lambda G(0) \in \pi(\Delta_p)$ if and only if $[1/w(z)]\exp[\lambda h(z)] \in H^p$. Each eigenspace is one-dimensional.*

Proof. (i): If $\lambda \in \pi(\Delta_p)$, we have from (2.4) that $\lambda / G'(b)$ is a nonnegative integer, which shows the first part of (i). Now suppose f is a nonzero solution of $\Delta_p(f) = kG'(b)f$, where k is a nonnegative integer. Then

$$f'(z) = \left(k\frac{h_0'(z)}{h_0(z)} - \frac{w'(z)}{w(z)} \right) f(z), \qquad \text{with} \quad h_0 = h \circ \gamma_b,$$

which has solutions $f(z) = [c/w(z)]h_0(z)^k$. The rest of part (i) follows easily from this.

(ii): We have

$$\Delta_p(f) = \frac{G(0)}{h'} f' + \frac{G(0)w'}{wh} f.$$

If $f(z) = [1/w(z)\exp[\lambda h(z)]$ is in H^p, then $\Delta_p(f) = \lambda G(0)f$, so $\lambda G(0) \in \pi(\Delta_p)$. Conversely, if $\Delta_p(g) = \lambda G(0)g$, where $g \in H^p$, $g \not\equiv 0$, then from (2.3) we see that

$$g(z) = c\frac{1}{w(z)}\exp[\lambda h(z)], \qquad c \neq 0.$$

Thus $[1/w(z)]\exp[\lambda h(z)] \in H^p$. The eigenspace corresponding to $\lambda G(0)$ is spanned by $[1/w(z)]\exp[\lambda h(z)]$. ∎

3. THE OPERATOR J_p.

We shall now work with a particular weighted composition semigroup to obtain information about the operator J_p defined by (1.4). Our results are as follows:

THEOREM 4. *For $1 < p < \infty$, J_p is a bounded linear mapping from H^p into H^p. Furthermore*:

(i) *If $1 < p \leqslant 2$ then $\|J_p\| = p/2(p-1)$.*
(ii) *If $2 < p < \infty$ then $\|J_p\| \leqslant p/2$.*
(iii) *If $1 < p < 2$, then the point spectrum of J_p is*

$$\pi(J_p) = \left\{\frac{1}{\lambda} : 2\left(1 - \frac{1}{p}\right) < \operatorname{Re}\lambda < \frac{2}{p}\right\}.$$

(iv) *If $2 \leqslant p < \infty$, then the point spectrum of J_p is empty.*
(v) *If $p = 2$, then the spectrum of J_2 is $\sigma(J_2) = \{z : |z - \frac{1}{2}| = \frac{1}{2}\}$.*

Proof. Let

$$S_t(f)(z) = \frac{1 - \varphi_t(z)}{1 - z} f(\varphi_t(z)), \qquad t \geqslant 0, \quad f \in H^p, \tag{3.1}$$

where

$$\varphi_t(z) = \frac{(1 + e^t)z + e^t - 1}{(e^t - 1)z + 1 + e^t}, \qquad z \in D, \quad t \geqslant 0. \tag{3.2}$$

Since

$$\frac{1 - \varphi_t(z)}{1 - z} = \frac{2}{(e^t - 1)z + 1 + e^t},$$

we see that condition (C_1) of Theorem 1 is satisfied. The semigroup (3.2) has generator $G(z) = \frac{1}{2}(1 + z)(1 - z)$, so by Theorem 2 the infinitesimal generator Δ_p of $\{S_t\}$ on H^p is given by

$$\Delta_p(f)(z) = \tfrac{1}{2}(1 + z)(1 - z)f'(z) - \tfrac{1}{2}(1 + z)f(z) \tag{3.3}$$

for each $f \in \mathscr{D}(\Delta_p)$.

Claim 1. For $t \geqslant 0$, the norm of S_t as an operator on H^p satisfies:

(a) $\|S_t\| \leqslant e^{-(1-1/p)t}$ if $1 < p < 2$;
(b) $\|S_t\| \leqslant e^{(-1/p)t}$ if $2 \leqslant p < \infty$.

Proof of claim 1: We denote by π^+ the right half plane, and by $H^p(\pi^+)$ the corresponding Hardy space. From [7, pp. 130–131] the linear mapping

$$V(f)(z) = \pi^{1/p} 2^{2/p} (1-z)^{-2/p} f(\mu(z)), \qquad f \in H^p(\pi^+), \quad z \in \mathbb{D},$$

$$\mu(z) = \frac{1+z}{1-z},$$

is an isometry from $H^p(\pi^+)$ onto H^p. For $t \geqslant 0$, let

$$\tilde{S}_t = V^{-1} \circ S_t \circ V,$$

where V^{-1}, the inverse of V, is given by

$$V^{-1}(f)(z) = \pi^{-1/p}(1+z)^{-2/p} f(\mu^{-1}(z)), \qquad f \in H^p, \quad z \in \pi^+,$$

$$\mu^{-1}(z) = \frac{z-1}{z+1}.$$

For $f \in H^p(\pi^+)$ and $t \geqslant 0$, a computation gives

$$\tilde{S}_t(f)(z) = \left(\frac{1+z}{1+e^t z} \right)^{1-2/p} f(e^t z). \tag{3.4}$$

If $z \in \pi^+$, then

$$e^{-t} \leqslant \left| \frac{1+z}{1+e^t z} \right| < 1. \tag{3.5}$$

Suppose first $2 \leqslant p < \infty$. From (3.4) and (3.5) we find

$$|\tilde{S}_t(f)(z)| \leqslant |f(e^t z)| \qquad \text{for each} \quad z \in \pi^+,$$

so

$$\begin{aligned}
\|\tilde{S}_t(f)\|_p &= \sup_{0<x<\infty}\left\{\int_{-\infty}^{\infty}|\tilde{S}_t(f)(x+iy)|^p\,dy\right\}^{1/p}\\
&\leqslant \sup_{0<x<\infty}\left\{\int_{-\infty}^{\infty}|f(e^tx+ie^ty)|^p\,dy\right\}^{1/p}\\
&\leqslant e^{(-1/p)t}\sup_{0<u<\infty}\left\{\int_{-\infty}^{\infty}|f(u+iv)|^p\,dv\right\}^{1/p}\\
&= e^{(-1/p)t}\|f\|_p,
\end{aligned}$$

where the second inequality is obtained by a change of variables in the integral. Thus $\|\tilde{S}_t\| \leqslant e^{(-1/p)t}$, and since V is an isometry, the conclusion follows.

Next suppose $1 < p < 2$; then $1-2/p < 0$. From (3.4) and (3.5) we now obtain $|\tilde{S}_t(f)(z)| \leqslant e^{(2/p-1)t}|f(e^tz)|$ for each $z \in \pi^+$. Repeating the argument for estimating $\|\tilde{S}_t\|$ as in the previous case, we find $\|S_t\| \leqslant e^{(2/p-1)}e^{(-1/p)t} = e^{-(1-1/p)t}$.

Claim 2. The point spectrum of Δ_p is $\pi(\Delta_p) = \{z: -1/p < \operatorname{Re} z < 1/p - 1\}$ for $1 < p < 2$, and it is empty for $p \geqslant 2$.

Proof of claim 2: The semigroup $\{\varphi_t\}$ in (3.2) has DW point $b = 1$. The associated univalent function h is given by $h(z) = \int_0^z [G(0)/G(\zeta)]\,d\zeta$, where G is the infinitesimal generator of $\{\varphi_t\}$ (see proof of Theorem 2). Since $G(z) = \frac{1}{2}(1+z)(1-z)$, we find

$$h(z) = \frac{1}{2}\log\frac{1+z}{1-z}.$$

Applying Theorem 3(ii) with $w(z) = 1-z$, we see that

$$\lambda \in \pi(\Delta_p) \quad \text{if and only if} \quad \frac{1}{1-z}\left(\frac{1+z}{1-z}\right)^{\lambda} \in H^p.$$

An easy argument shows that $(1+z)^{\lambda}(1-z)^{-\lambda-1}$ is in H^p if and only if both $(1+z)^{\lambda}$ and $(1-z)^{-\lambda-1}$ are in H^p. The latter is equivalent to $\operatorname{Re}\lambda > -1/p$ and $\operatorname{Re}\lambda < 1/p - 1$. Thus $\lambda \in \pi(\Delta_p)$ if and only if $-1/p < \operatorname{Re}\lambda < 1/p - 1$. This finishes the proof of claim 2.

We finish now the proof of the theorem. From claim 1, the type $\omega_0 = \lim_{t\to\infty}(\log\|S_t\|)/t$ of $\{S_t\}$ satisfies $\omega_0 \leqslant -1/p$ for $2 \leqslant p < \infty$ and

$\omega_0 \leqslant -(1-1/p)$ for $1<p<2$. In either case, from [4, Theorem VIII.1.11] we see that $z=0$ is in the resolvent set of Δ_p. A calculation with power series shows that the resolvent function of Δ_p at 0 is

$$R(0, \Delta_p) = 2J_p, \qquad 1<p<\infty.$$

This shows that J_p is bounded on H^p. From the spectral mapping theorem and claim 2 we obtain parts (iii) and (iv) of the theorem. It follows that $\|J_p\| \geqslant p/[2(p-1)]$ for $1<p<2$. Applying the Hille-Yosida-Phillips theorem in the form of [4, Corollary VIII.1.14], we obtain (in view of claim 1) $\|J_p\| \leqslant p/2$ for $2 \leqslant p < \infty$ and $\|J_p\| \leqslant p/[2(p-1)]$ for $1<p<2$. This shows that $\|J_p\| = p/[2(p-1)]$ for $1<p<2$.

Assume finally that $p=2$. Observe that $[1-\varphi_t(z)]/(1-z) = e^{(-1/2)t}[\phi_t'(z)]^{1/2}$. Furthermore each φ_t is a (hyperbolic) Möbius transformation of $\mathbb{D}$, so from [6, Theorem 2] we have that $e^{\frac{1}{2}t}S_t$ is an isometry of H^2 for each t. Also from [1, Theorem 3.1(iii)] the spectrum of the infinitesimal generator of $\{e^{\frac{1}{2}t}S_t\}$ is the imaginary axis. It follows that the spectrum of Δ_2 is $\sigma(\Delta_2) = \{-\frac{1}{2}+i\lambda : \lambda \in \mathbb{R}\}$; hence $\sigma(J_2) = \{z : |z-\frac{1}{2}| = \frac{1}{2}\}$. Since also $\|J_2\| \leqslant 1$ as proved earlier, we have $\|J_2\| = 1$. This finishes the proof. ■

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Received 14 October 1985; revised 4 February 1986